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On the treatment of the gravitational field in connection with the generally relativistic Dirac equation

By O. Klein

In a recent paper R. Arnowitt and S. Deser[1] have successfully applied the so-called Palatini method, i.e. treating the g_{ik} and the Γ^i_{kl} as independent variables to the quantization of the gravitational field. In this connection it occurred to me that a further simplification of the simultaneous quantum-theoretical treatment of the gravitational field and a Fermion field may be obtained if the coordinate dependent γ^i and the Γ_i appearing in the generally relativistic formulation of the Dirac equation are treated as independent quantities. This may be done by means of the Feynman method, taking the quantities or rather their components with respect to a fixed set of Dirac matrices as integration variables of the Feynman functional integral. In the following, however, we shall limit ourselves to the variation principle treatment of the gravitational field alone by means of the quantities in question.

In the usual treatment of the Dirac equation in general relativity theory[2] the g_{ik} are connected with the coordinate-dependent Dirac matrices γ_i by means of the anticommutation relations

$$g_{ik} = \tfrac{1}{2}\{\gamma_i, \gamma_k\} \tag{1}$$

which we shall take, now, as the definition of the metric tensor g_{ik}. Further, the Γ_k are defined by means of the equations

$$\gamma_{i,k} = [\Gamma_k, \gamma_i] \tag{2}$$

where

$$\gamma_{i,k} = \frac{\partial \gamma_i}{\partial x^k} - \Gamma^r_{ik}\gamma_r \tag{3}$$

are the ordinary covariant derivatives of γ_i. Further, the Γ_k are connected with the Riemann curvature tensor $R_{kl,mn}$ by means of

$$[\Phi_{kl}, \gamma_m] = R_{kl,mn}\gamma^n \tag{4}$$

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[1] R. Arnowitt and S. Deser, Phys. Rev. *113*, 745 (1959).

[2] See V. Bargmann, Berl. Ber. 1932, p. 346.

where
$$\Phi_{kl}=\frac{\partial \Gamma_l}{\partial x^k}-\frac{\partial \Gamma_k}{\partial x^l}-[\Gamma_k,\, \Gamma_l]. \tag{5}$$

With
$$s^{mn}=\tfrac{1}{2}[\gamma^m,\, \gamma^n] \tag{6}$$

it follows in a wellknown way, using (1), that

$$\Phi_{kl}=-\tfrac{1}{4} R_{kl,\, mn}\, s^{mn} \tag{7}$$

satisfies (4) and is the only solution of that equation, if to (2) we add the usual convention
$$\mathrm{Tr}\,(\Gamma_k)=0. \tag{8}$$

We consider, next, the scalar

$$\Phi=\tfrac{1}{2}\Phi_{kl}\, s^{kl}=-\tfrac{1}{8} R_{kl,\, mn}\, s^{kl}\, s^{mn}. \tag{9}$$

Let us put
$$\gamma^k=\gamma^k_{\cdot r}\, e^r,\ \{e^r,\, e^s\}=2\,\delta_{rs},\ g^{kl}=\gamma^k_{\cdot r}\, \gamma^l_{\cdot r} \tag{10}$$

Then
$$\Phi=-\tfrac{1}{8} R_{kl,\, mn}\, \gamma^k_{\cdot p}\, \gamma^l_{\cdot q}\, \gamma^m_{\cdot r}\, \gamma^n_{\cdot s}\, e^p\, e^q\, e^r\, e^s. \tag{11}$$

As is easily proved

$$\mathrm{Tr}\,(e^p\, e^q\, e^r\, e^s)=4\,(\delta_{pq}\,\delta_{rs}-\delta_{pr}\,\delta_{qs}+\delta_{ps}\,\delta_{qr}) \tag{12}$$

so that

$$\begin{aligned}\mathrm{Tr}\,(\Phi)&=-\tfrac{1}{2} R_{kl,\, mn}\, \gamma^k_{\cdot p}\, \gamma^l_{\cdot q}\, \gamma^m_{\cdot r}\, \gamma^n_{\cdot s}\,(\delta_{pq}\,\delta_{rs}-\delta_{pr}\,\delta_{qs}+\delta_{ps}\,\delta_{qr})=\\ &=-\tfrac{1}{2} R_{kl,\, mn}\,(g^{kl}\, g^{mn}-g^{km}\, g^{ln}+g^{kn}\, g^{lm})=\\ &=\tfrac{1}{2}(R^{mn}_{mn}-R^{nm}_{mn}),\ \text{or}\ \mathrm{Tr}\,(\Phi)=R.\end{aligned} \tag{13}$$

Let us, now, consider the action integral for the gravitational field extended over a given space-time region

$$s=\int \sqrt{g}\, d^4 x\, R \tag{14}$$

whereby for R we write $\mathrm{Tr}\,(\Phi)$, using the expression (9) for Φ together with (5) und (6) by means of which Φ is expressed in the γ^k and the Γ_k and their first space-time derivatives, g being the absolute value of the determinant $|g_{ik}|$. Let us, now, carry out a variation of the action integral s with respect to the Γ_k and γ^k, assuming that their variations vanish at the boundary of the region. After partial integration and making use of the variation of a trace being the trace of the variation and a trace of a product AB being independent of the order of the factors A and B we get

$$\delta s = \mathrm{Tr} \int \sqrt{g}\, d^4 x \left\{ \left(\frac{1}{\sqrt{g}} \frac{\partial \sqrt{g}\, s^{kl}}{\partial x^l} + [s^{kl}, \Gamma_l] \right) \delta \Gamma_k - \tfrac{1}{2} \left([\Phi_{kl}, \gamma^l] + \{\Phi, \gamma_k\} \right) \delta \gamma^k \right\}.$$

Let us, next, consider the field equations obtained by putting the variation in question equal to zero for arbitrary $\delta \Gamma_k$ and $\delta \gamma^k$. As is easily seen, the sufficient and necessary conditions for this are the following two sets of equations

$$\frac{1}{\sqrt{g}} \frac{\partial \sqrt{g}\, s^{kl}}{\partial x^l} + [s^{kl}, \Gamma_l] = 0 \quad \text{and} \quad [\Phi_{kl}, \gamma^l] + \{\Phi, \gamma_k\} = 0 \tag{15}$$

which on account of (5) and (6) form eight first order differential equations for the four γ^k and the four Γ_k. From general reasons of covariance it would seem that they are equivalent to the Einstein field equations in the absence of matter.

Leaving this question aside we shall at present limit ourselves to some remarks on the first set of equations (15), which may be given the form

$$s^{kl}{}_{,l} + [s^{kl}, \Gamma_l] = 0. \tag{16}$$

As is easily seen, this relation is an immediate consequence of (2) on account of (6) and the vanishing of the covariant derivative of g_{ik}. In fact, if, as is usual in the study of the generally relativistic wave equation, we define a generalized covariant derivative $A_{;k}$, A being any quantity, by means of

$$A_{;k} = A_{,k} + [A, \Gamma_k]. \tag{17}$$

Then

$$(AB)_{;k} = A_{;k} B + A B_{;k} \tag{18}$$

of which the statement in question is an immediate consequence.

On the other hand, relation (16) is weaker than (2), which together with the assumption $\mathrm{Tr}(\Gamma_k) = 0$ determines Γ_k uniquely. Since any four-vector Λ_k satisfying the conditions

$$[s^{kl}, \Lambda_l] = 0 \tag{19}$$

added to a solution of (16) gives a new solution we have to look for such four-vectors. Now, the quantity

$$\gamma = \frac{\sqrt{g}}{24} \sum_{k,l,m,n} \varepsilon_{klmn} \gamma^k \gamma^l \gamma^m \gamma^n,$$

where ε_{klmn} is the Levi-Civita antisymmetric tensor density of weight -1, is seen to commute with all the s^{kl} and is, moreover, invariant with respect to general coordinate transformations. Thus,

$$\Gamma'_k = \Gamma_k + a_k + \gamma\, b_k \tag{20}$$

where Γ_k may be taken to be that solution of (2), which has the trace equal to zero, and where a_k and b_k behave as vectors in general coordinate transformations, would seem to be the most general solution to (15). Since in the special relativistic case $\gamma = \gamma_5$ the b_k form a pseudovector in the usual notation, while the a_k form an ordinary vector. Hereby, since $\mathrm{Tr}\,(\gamma) = 0$, only the a_k-term is included by the condition $\mathrm{Tr}\,(\Gamma_k) = 0$.[1]

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Institute for Theoretical Physics, University of Stockholm.

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[1] B. Laurent (Arkiv för Fysik *16*, p. 278, 1960) has made a thorough study of the theory, only sketched above, with very interesting results including the field equations in the presence of matter and the conservation theorems.

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